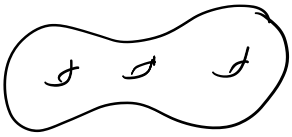
An Introduction to Mapping Class Groups

Ref A primer on mapping class group (Farb and Margalit)

$S = S_g$ closed, orientable surface



$MCG(S) = \mathrm{Homeo}^+(S) / \mathrm{Homeo}^+_0(S)$ orientation-preserving homeomorphisms that are isotopic to the identity (easy to check that it's a normal subgroup)

Examples of mapping class groups

(Alexander's trick) Let $f \in \mathrm{Homeo}^+(D^n, \mathrm{rel}\, \partial)$ $f|_{\partial D^n} = id$

Then f is isotopic to the identity.

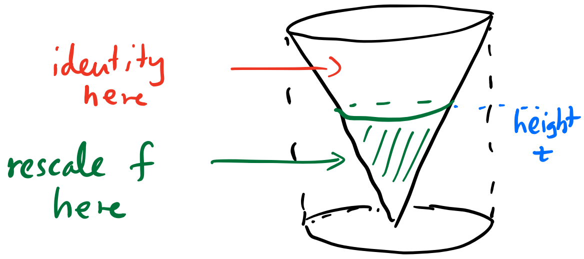
a) **Corollary** $MCG(D^2) = \{1\}$.

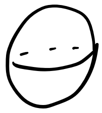
proof of Alexander's Lemma

Let $f : D^n \to D^n$ s.t. $f|_{\partial D^n} = id_{\partial D^n}$. Define an isotopy between f and id_{D^n} by the formula

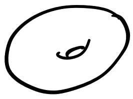
$$H(x,t) = \begin{cases} t\, f(x/t) & \text{if } 0 \leqslant \|x\| < t \\ x & \text{if } t \leqslant \|x\| \leq 1 \end{cases}$$

Exercise Check that H is continuous at $t = 0$



b) $g=0$  $MCG(\text{Sphere}) = \{1\}$

proof: First isotope $f \in Homeo^+(S^2)$ so that f fixes a point p. Identify $S^2 \setminus \{p\}$ with $\mathbb{R}^2$, and use the linear isotopy in $\mathbb{R}^2$ to show that f is isotopic to the identity.

c) $g=1$  $MCG(\text{torus}) \cong SL(2,\mathbb{Z})$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

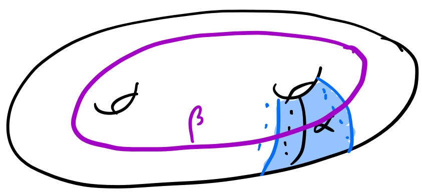
$A: \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$

Remark: Mapping class groups of higher genus surfaces don't have a simple description, although they have explicit finite presentations.

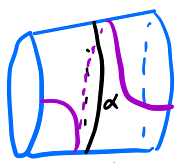
Examples of mapping classes

Dehn twist about α

α: simple closed curve



T_α: right-handed twist



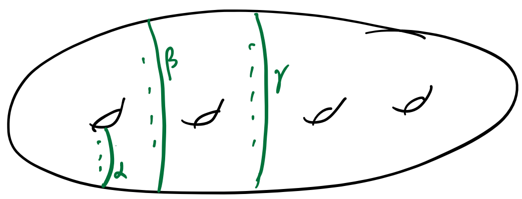
$T_\alpha(\beta)$

Say that simple closed curves α, β are of the same type if there is a homeomorphism ϕ of S s.t. $\phi(\alpha) = \beta$.

Exercise Show that there are only finitely many types of simple closed curves on a given compact surface.

(Hint: Use the classification of surfaces)

Example



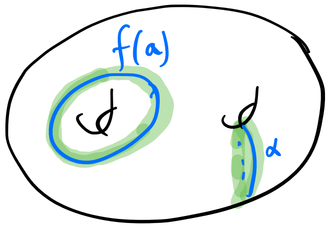
α : non-separating

β, γ : separating

β can be distinguished from γ since β decomposes the surface differently than γ.

Fact Suppose α, β are simple closed curves such that $T_\alpha = T_\beta$. Then $\alpha = \beta$.

Fact Let $f \in MCG(S)$, and α be a simple closed curve. Then $T_{f(a)} = f\, T_a\, f^{-1}$.



In particular, if α and β have the same type, then T_α and T_β are conjugate in the mapping class group.

For examples, Dehn twists around any two non-separating simple closed curves are conjugate in the mapping class group.

Later, we use this fact to compute abelianisation of the mapping class group.

Theorem For $g \geq 3$, the center of $MCG(S_g)$ is trivial.

proof suppose $f \in Z(MCG(S_g))$. We first show that
↑ center

f should fix the isotopy class of every simple closed curve.

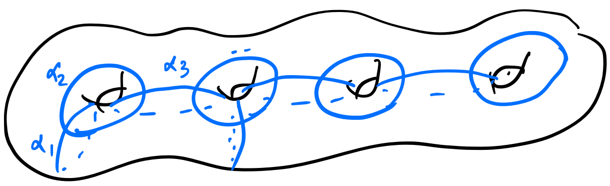
Let α be a simple closed curve and T_α be the Dehn twist about α. We have

$$T_{f(\alpha)} = f T_\alpha f^{-1} = T_\alpha$$

↑ since $f \in Z(MCG(S_g))$

$\Rightarrow$ α is isotopic to $f(\alpha)$.

Now consider the set of simple closed curves shown below:



With some work (see proposition below) one can deduce that after an isotopy, f fixes each vertex and each oriented edge of the graph $\bigcup_i \alpha_i$. Note that each complementary region to this graph is a disc. (we say that $\{\alpha_i\}$ fill S.) By Alexander's trick, the restriction of f to each complementary disc region is isotopic to identity. Therefore f is isotopic to identity.

$\square$

proposition (Alexander method)

Let S be a compact surface and ϕ be an orientation-preserving homeo. of S fixing the boundary pointwise. Let $\gamma_1, \dots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S s.t.

1) The γ_i are pairwise in minimal position.

2) The γ_i are pairwise non-isotopic.

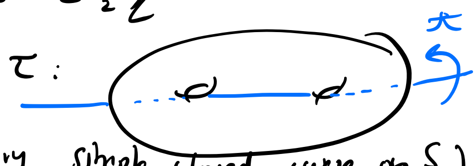
3) For distinct i, j, k at least two of $\gamma_i, \gamma_j, \gamma_k$ are disjoint from each other.

a) If there is a permutation σ of $\{1, 2, \dots, n\}$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\bigcup \gamma_i)$ is isotopic to $\bigcup \gamma_i$ relative to ∂S.

If we consider $\bigcup \gamma_i$ as a graph Γ in S, then the composition of ϕ with this isotopy gives an automorphism ϕ_* of the graph Γ.

b) suppose that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ, with orientations, then ϕ is isotopic to the identity. Otherwise ϕ has a non-trivial power that is isotopic to the identity.

Remark The center of $\mathrm{Mod}(S_2)$ is $\mathbb{Z}/_2\mathbb{Z}$ and it is generated by the hyperelliptic involution τ :



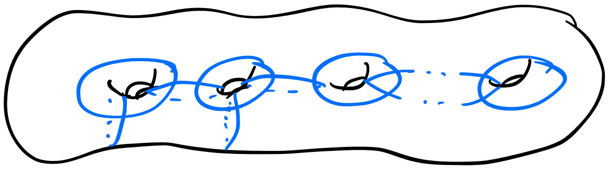
(Note that τ fixes the isotopy class of every simple closed curve on S_2)

Exercise Use the relation $(T_{f(a)})^n = f(T_a)^n f^{-1}$ to show that every finite index subgroup of $\mathrm{Mod}(S_g)$ for $g \geq 3$ has trivial center.

Theorem (Dehn–Lickorish)

The group $\mathrm{Mod}(S_g)$ is generated by finitely many Dehn twists about non-separating simple closed curves.

Humphries generating set



We only prove the weaker statement

Theorem $MCG(S_g)$ is generated by the (infinite collection) of Dehn twist about all non-separating curves.

proof We prove the statement more generally for any compact orientable surface $S_{g,n}$. The proof is by induction on g and n (more specifically on lexicographic pairs (g,n))

Case 1 $g \geq 2$. Let $f \in \mathrm{Mod}(S_{g,n})$. Let a be a non-separating simple closed curve. We show that there is a product h of Dehn twists about non-sep s.c.c. that takes $f(a)$ to a.$^{(*)}$ Then hf fixes the curve a, and we can use the cutting homomorphism below to induct:

$$1 \to \langle T_a \rangle \to \boxed{\mathrm{Mod}(S, a)} \longrightarrow \mathrm{Mod}(S-a) \to 1$$

hf fixes a and so belongs here.

surface obtained by cutting S along a has signature $(g-1, n+2)$

<u>case 2</u> $g \geq 1$, $n \geq 1$. We have the forgetful map

$$\mathrm{Mod}(S_{g,n}) \twoheadrightarrow \mathrm{Mod}(S_{g,n-1})$$

obtained by forgetting a puncture. The kernel of this map is the set of mapping classes obtained by 'pushing' the n-th puncture around the surface. The kernel is isomorphic to $\pi_1(S_{g,n-1})$, and we have the Birman exact sequence

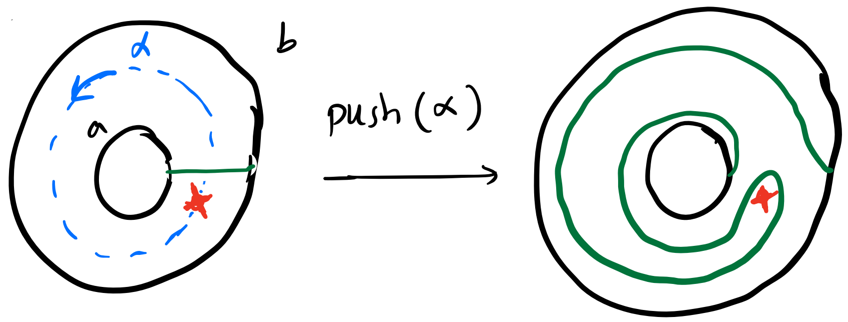
point pushing subgroup of mapping class group

$$1 \to \pi_1(S_{g,n-1}) \to \mathrm{Mod}(S_{g,n}) \to \mathrm{Mod}(S_{g,n-1}) \to 1$$

By induction hypothesis, $\mathrm{Mod}(S_{g,n-1})$ is generated by Dehn twists about non-sep s.c.c.. We also show that the kernel $\pi_1(S_{g,n-1})$ is generated by Dehn twists about non-sep s.c.c. to complete the proof.

<u>Fact</u> Let α be a s.c.c. on S. Then $\mathrm{push}([\alpha]) = T_a T_b^{-1}$ where a, b are s.c.c. in S obtained by pushing α off itself to the left and right, respectively.

class of α in $\pi_1(S, x)$



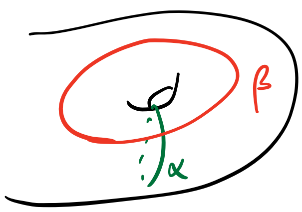
It remains to show that (*) holds: Let $f(\alpha) = \beta$.

Then (*) is equivalent to:

{Given non-sep s.c.c. α, β, there is a product of Dehn twists about non-sep s.c.c. taking β to α.}

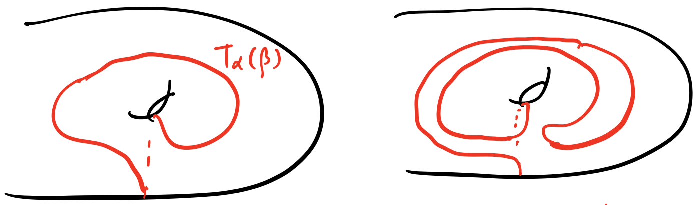
Special case of (*): Assume that $i(\alpha, \beta) = 1$

geometric int. number



In this case, we can explicitly see that

$$T_\beta T_\alpha(\beta) = \alpha$$



$T_\beta T_\alpha(\beta) = \alpha$

General case

It can be shown by induction on $i(\alpha, \beta)$ that there is a sequence of non-sep s.c.c. $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k = \beta$ such that for each $1 \leq j \leq k-1$ we have

$$i(\alpha_j, \alpha_{j+1}) = 1.$$

$$\alpha=\alpha_1 \underset{\text{product of non-sep twists}}{\overset{\text{special case}}{\curvearrowright}} \alpha_2 \underset{\text{product of non-sep twists}}{\overset{\text{special case}}{\curvearrowright}} \alpha_3 \cdots \underset{\text{product of non-sep twists}}{\overset{\text{special case}}{\curvearrowright}} \alpha_K=\beta$$

This completes the proof that $\mathrm{Mod}(S_g)$ is generated by Dehn twists about non-sep s.c.c.. $\square$

__Remark__ Define a graph whose vertices are isotopy classes of non-sep s.c.c. on S, and with an edge between two vertices if the corresponding curves have geometric intersection number 1. Then we stated that this graph, called the non-separating curve graph, is connected.

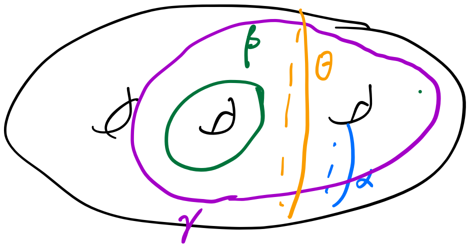
Another important variant is the curve graph defined as follows:

vertices: isotopy classes of s.c.c. on S
edges: disjointness (geometric int. number 0)

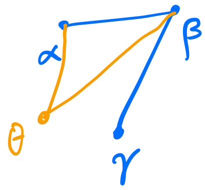
It can be shown that the curve graph is connected as well. Since $\mathrm{Mod}(S)$ acts on the curve graph $\mathcal{C}(S)$, the curve graph can be used to study the mapping class group.

__Exercise__ What are the stabilisers of vertices and edges of the curve graph for the action of the mapping class group?

curve graph



edge: disjointness



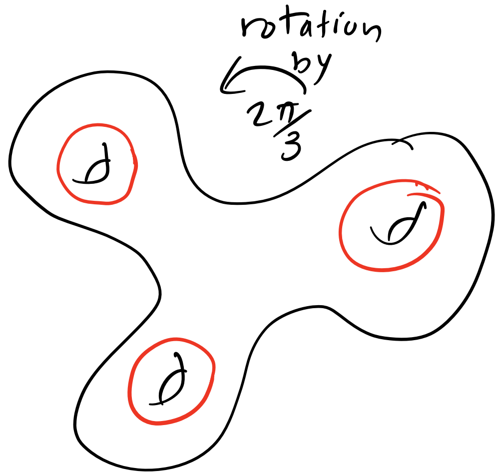
Nielsen – Thurston classification for MCG

A mapping class f is called <u>periodic</u> if it is of finite order. This means that there is $n \in \mathbb{N}$ such that f^n is isotopic to the identity. By a theorem of Nielsen f is periodic iff f is isotopic to a periodic homeomorphism.

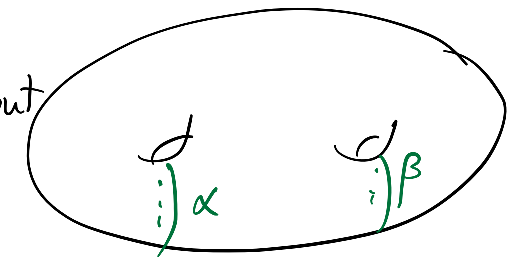
A mapping class is called <u>reducible</u> if there is a non-empty collection $\mathcal{C}$ of disjoint essential simple closed curves in S such that $f(\mathcal{C}) = \mathcal{C}$.

examples

1) ρ: periodic and reducible

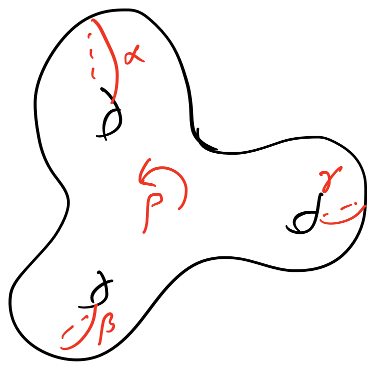


T_α = Dehn twist about α



2) $T_\alpha T_\beta$ is reducible

3) $\rho \circ T_\alpha \circ T_\beta \circ T_\alpha$ reducible but not periodic



$\mathcal{C} = \{\alpha, \beta, \gamma\}$

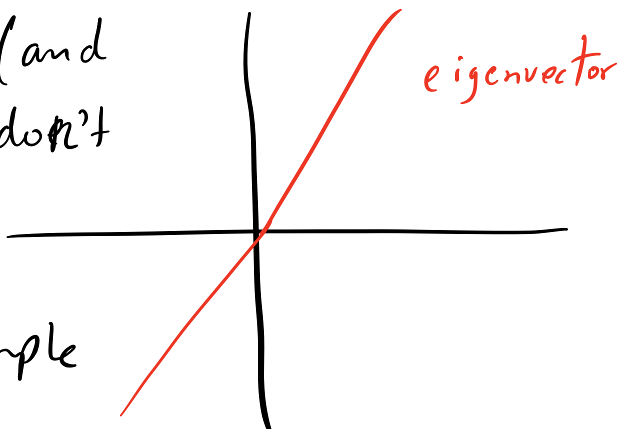
A mapping class f is called <u>pseudo-Anosov</u> if no power of f fixes the isotopy class of an essential simple closed curve.

<u>example</u> $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$

The eigenvalues of A are roots of $x^2 - 3x + 1 = 0$

In particular the eigenvalues are irrational and the eigenvectors have irrational slopes.

It follows that A (and any of its powers) don't fix the isotopy class of any essential simple closed curve.



A is an <u>Anosov map</u>.

pseudo-Anosov maps are generalisation of Anosov maps of a two-dim torus.

<u>(Nielsen–Thurston classification)</u>

Let S be a closed orientable surface of genus $g \geq 1$. Then every mapping class is either

i) periodic; or

ii) reducible; or

iii) pseudo-Anosov.

Moreover, iii) is disjoint from i) and ii).